Reduction of Order Examples

1. Solve $t^2y'' - 4ty' + 6y = 0$, $y_1 = t^2$

Soln:

Let $y_2 = vy_1$

$t^2(vy_1)'' - 4t(vy_1)' + 6vy_1 = 0$

$t^2(v''y_1 + 2v'y_1' + vy_1'') - 4t(v'y_1 + vy_1')$

$+ 6vy_1 = 0$

$t^2v''y_1 + 2t^2v'y_1' + vt^2y_1'' - 4tv'y_1 - 4tvy_1'$

$+ 6vy_1 = 0$

Collect all the terms with v.

$v(\underbrace{t^2y_1'' - 4ty_1' + 6y_1}) = 0$

Since y_1 is a soln to the original eqn, this equals 0.

$t^2v''y_1 + 2t^2v'y_1' - 4tv'y_1 = 0$

$v''t^4 + 4v't^3 - 4v't^3 = 0$

$v''t^4 = 0$

$v'' = 0$

$v' = \int 0 \, dt$

$= c \leftarrow$ C is a constant.

$v = \int c \, dt$

$= ct + c_1 \leftarrow$ c and c_1 are constants.

Let c and c_1 equal 1 and 0, respectively.

$v = t$

$y_2 = vy_1$

$= t^3$

2. Solve $t^2 y'' + 2ty' - 2y = 0$, $y_1 = t$

Soln:

Let $y_2 = vy_1$

$t^2(vy_1)'' + 2t(vy_1)' - 2vy_1 = 0$

$t^2(v''y_1 + 2v'y_1' + vy_1'') + 2t(v'y_1 + vy_1') - 2vy_1 = 0$

$t^2v''y_1 + 2t^2v'y_1' + t^2vy_1'' + 2tv'y_1 + 2tvy_1' - 2vy_1 = 0$

Collect all the terms with v.

$v(t^2y_1'' + 2ty_1' - 2y_1) = 0$

$t^2v''y_1 + 2t^2v'y_1' + 2tv'y_1 = 0$

$v''t^3 + 2t^2v' + 2t^2v' = 0$

$v''t^3 + 4t^2v' = 0$

$v''t + 4v' = 0$

Let $w = v'$ and $w' = v''$

$t\dfrac{dw}{dt} + 4w = 0$

$t\,dw = -4w\,dt$

$\dfrac{-1}{4w}\,dw = \dfrac{1}{t}\,dt$

$\displaystyle\int \frac{-1}{4w}\,dw = \int \frac{1}{t}\,dt$

$\dfrac{-\ln|w|}{4} + c_1 = \ln|t| + c_2$

$\dfrac{-\ln|w|}{4} = \ln|t| + c_2$

$\ln|w| = -4\ln|t| + c_2$

$w = e^{-4\ln|t| + c_2}$

$= (e^{\ln|t|})^{-4} \cdot e^{c_2}$

$= c'(\quad t)^{-4}$

Let $c' = 1$

$w = t^{-4}$

Recall $v' = w$

$v = \int w \, dt$

$= \int t^{-4} \, dt$

$= -3t^{-3} + c_1$

Let $c_1 = 0$

$v = -3t^{-3}$

$y_2 = v y_1$

$= (-3t^{-3})(t)$

$= -3t^{-2}$